Regeneration Theory and Experiment *

By

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A comprehensive criterion for the stability of linear feed-back circuits has recently been formulated by H. Nyquist, in terms of the transfer factor around the feed-back loop. The importance of any such general criterion lends interest to an experimental verification, with which the paper is primarily concerned.

The subject is dealt with under five principal headings. The first section reviews some of the criteria for oscillation to be found in the literature of vacuum tube oscillators. The second describes the derivation of Nyquist's criterion somewhat along the lines followed by Routh in one of his investigations of the stability of dynamical systems. The third part deals with two experimental methods used in measuring the transfer factor. The fourth is concerned with the particular amplifier circuit used in the test of Nyquist's criterion. The last section applies the criterion to a nonlinear case, and to circuits including two-terminal negative impedance elements.

In a comparatively recent paper on "Regeneration Theory," [1] Dr. Nyquist presented a mathematical investigation of the conditions under which instability [2] exists in a system made up of a linear amplifier and a transmission path connected between its input and output circuits. The results of the investigation are of interest because of their obvious application to amplifiers provided with feed-back paths,[3] as well as to the starting conditions in oscillators. As a result of his general analysis, Dr. Nyquist arrived at a criterion for stability, expressed in particularly simple and convenient form, which is not restricted in its range of application to particular amplifier and circuit configurations.

The great value attached to a criterion as precise and as general as Nyquist's makes it desirable to submit the criterion to an experimental test. One particularly striking conclusion drawn from this criterion is that under certain conditions a feed-back amplifier may sing within certain limits of gain, but either reduction or increase of gain beyond these limits may stop singing. A feed-back amplifier satisfying these conditions was set up, and the experimental results were found to be in agreement with this conclusion.

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[1] *Bell. Sys. Tech. Jour.*, vol. XI, p. 126.

[2] Instability is used in the sense that a small impressed force, which dies out in course of time, gives rise to a response which does not die out.

[3] *Electrical Engineering*, July, 1933; *Bell Sys. Tech. Jour.*, p. 258, July, 1933.

It is interesting to compare the criterion with those derived for the mechanical systems of classical dynamics. In his Adams Prize Paper on "The Stability of Motion," [4] and again in his "Advanced Rigid Dynamics," [5] Routh investigated the general problem of dynamic stability and established a number of criteria based upon various properties of dynamical systems. When applied to the problem of feed-back amplifiers, keeping Nyquist's result in mind, one of them is found to be equivalent to Nyquist's criterion, although expressed in different terms and derived in a different way.

To provide a background for the experiments, we propose to state some of the criteria for stability which are to be found in the literature of vacuum tube oscillators, and to compare them with Nyquist's or Routh's criterion, the development of which is most conveniently described somewhat along the lines followed by Routh. Following this we shall deal with the experimental methods and apparatus which were used in testing the criterion, and conclude with some extensions of the criterion.

Circuit Analysis and Stability

Conditions required for the starting of oscillations in linear feed-back circuits, corresponding to instability, are to be found in the literature of vacuum tube oscillator circuits, expressed in a number of ostensibly different forms. These are usually based upon the familiar mesh differential equations for the system which involve differentiations and integrations of the mesh amplitudes with respect to time. Using the symbol p to denote differentiation with respect to time, each mesh equation becomes formally an algebraic one in p, involving the circuit constants and the mesh amplitudes. The solution of this system of equations is known to be expressible as the sum of steady state and transient terms. The transient terms are each of the form $B_k \epsilon^{p_k t}$, the B_k's being fixed by initial conditions, and the p_k's being determined from the circuit equations. If we set up the determinant of the system of equations—the discriminant—and equate it to zero, the roots of the resulting equation are the p_k's above. In general each mesh equation involves p to the second degree at most, and with n meshes the discriminant is of degree $2n$ at most. Accordingly we may express the determinantal equation as

$$F(p) = 0 = K(p - p_1)(p - p_2) \cdots (p - p_{2n}). \tag{1}$$

As for the steady state term, in the simplest case in which a sinusoidal wave of frequency $\omega/2\pi$ is impressed, it is equal to the impressed

[4] Macmillan, 1877.
[5] Macmillan, 6th edition, 1905.

voltage divided by the discriminant and multiplied by the appropriate minor of the determinant, in which p is replaced by $j\omega$. The character of the response due to a slight disturbance and in the absence of any periodic force is determined by the exponentials. In general, p_k is a complex quantity which may be written as $a_k + j\omega_k$. It is apparent that in the critical case for which a_k is zero, the corresponding term becomes $\epsilon^{j\omega_k t}$, corresponding to an oscillation invariable in amplitude, of frequency $\omega_k/2\pi$. If a_k is negative, as is ordinarily the case when the system is passive (containing no amplifier or negative impedance), then the oscillation diminishes in course of time. When a_k is positive, however, the oscillation increases with time, and the system is said to be unstable. Evidently the stability of a system is determined by the signs of the a_k's.

Several criteria which have previously been enunciated for the maintenance of free oscillations are deducible from the above. One states that the discriminant must vanish when p takes on the value $j\omega$. Another states that the damping (a_k) must be zero at the frequency of oscillation. These are clearly equivalent. Two derived criteria may also be mentioned, based upon the properties of the system when the circuit is broken. The first of these states that if the impedance is measured looking into the two terminals provided by the break. the impedance must be zero at the frequency of steady oscillation.

The second criterion involving the transfer factor has become fairly widespread, perhaps because it leads to a simple and plausible physical picture. To determine the transfer factor around the feed-back loop, the loop is broken at a convenient point, and the two sets of terminals formed by the break are each terminated in a passive impedance equal to that which is connected in the normal (unbroken) condition. Then when a voltage of frequency $\omega/2\pi$ is applied to one of the pairs of terminals so provided—the input terminals [6]—and the corresponding voltage is measured across the other pair, the transfer factor $A(j\omega)$ is obtained as the vector ratio of the output voltage to the input voltage.

The manner in which the transfer factor enters into the problem may be demonstrated directly by comparing the voltages at any point of the main amplifier circuit under the two conditions in which the feed-back path is opened and closed respectively. If with the feed-back path open the voltage at any such point is $E\epsilon^{pt}$, then when the feed-back path is closed the voltage will be changed [7] to

$$E\epsilon^{pt}/[1 - A(p)].$$

[6] Input terminals are those across which an impressed potential leads to propagation in the normal direction of amplifier transmission.

[7] *Bell Sys. Tech. Jour.*, Vol. XI, p. 128.

This may be shown as follows with reference to the particular circuit of Fig. 1: If the feed-back circuit is broken and then properly terminated, the voltage existing across the input is taken as e. Now suppose the feed-back path to be restored. Designating the voltage existing across the input in the presence of feed-back as e_1 we have $e_1 = e + Ae_1$, from which the above equation follows.

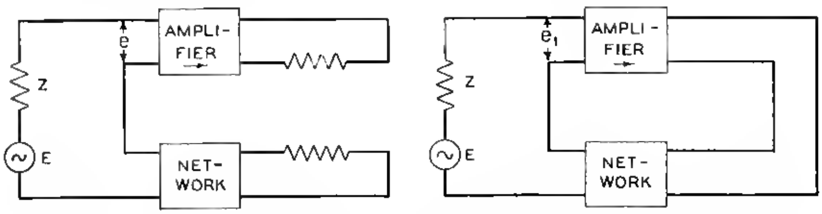

Fig. 1—Series type feed-back; loop broken and terminated at left, normal feed-back circuit at right.

If we let $F_1(p)$ represent the discriminant of the system when the loop is broken and terminated, then the roots of the equation formed by setting the discriminant equal to zero are assumed to have positive real parts. Now for the corresponding discriminant when the loop is restored, we have in accordance with the above considerations

$$F(p) = [(1 - A(p)]F_1(p).$$

In setting this discriminant equal to zero to obtain the roots, the only ones which have nonnegative real parts are those corresponding to the feed-back term

$$f(p) = 1 - A(p). \tag{2}$$

The above-mentioned criterion may be deduced from this expression. For steady oscillations to exist the output potential must be identical in amplitude and in phase with that existing across the input at the frequency of oscillation ($p = j\omega$), in which case the transfer factor is unity. This seems reasonable on the basis that when the input and output terminals are connected through, the oscillation will neither increase nor decrease with time. It may be demonstrated by direct analysis that these several criteria, framed for the critical case of undamped oscillations, all lead to the same correct conclusion.

Of course in any actual oscillating circuit it is practically impossible to get these conditions fulfilled exactly, and what is ordinarily done in the practical design of oscillating circuits is to ensure that the voltage fed back will be greater than that required to produce oscillation. This evidently goes a step further than the above criteria, and reliance is placed upon the nonlinear properties of the circuit to fulfill the criteria automatically. The procedure is known by experiment to be effective in the usual type of oscillating circuit. In particular forms of feed-back circuits, however, it may be demonstrated that *the transfer factor may be made greater than unity without giving rise to oscillations.* This situation was investigated experimentally, and found to be in accord with the stability criterion stated by Nyquist.

Nyquist's Criterion

The explicit solution of (1) for the p_k's demands an exact knowledge of the configuration of the amplifier and feed-back circuits. When the number of meshes is large, the solution involves much labor. If we wish simply to observe whether or not the system is stable, however, we need not obtain explicit solutions for the roots; in fact, all we need to know is whether or not any one of the p_k's has its real part positive. It turns out that when we know the transfer factor as a function of frequency, by calculation or by measurement, a simple inspection of the transfer factor polar diagram suffices for this purpose. This diagram is constructed by plotting the imaginary part of the transfer factor against the real part for all frequencies from minus to plus infinity.[8]

To obtain Nyquist's criterion we consider the vector drawn from the point (1, 0) to a point moving along the polar diagram; if the net angle which the vector swings through in traversing the curve is zero, the system is stable; if not, it is unstable. To express it in the terms used by Routh, if we set $1 - A(j\omega) = P + jQ$, and observe the changes of sign which the ratio P/Q makes when P goes through zero as the frequency steadily increases, the system is stable when there are the same number of changes from plus to minus as from minus to plus. It may be demonstrated that these two statements are equivalent.

The way in which the above procedures may be shown to reveal the existence of a root with positive real part may be outlined somewhat along the lines followed by Routh in his analysis.[9] Since p is a com-

[8] The transfer factor for negative frequencies $A(-j\omega)$ is the complex conjugate of that for positive frequencies $A(j\omega)$. Thus, if

$$A(j\omega) = X + jY,$$

then

$$A(-j\omega) = X - jY.$$

[9] A number of restrictions on the generality of the analysis may be noted. It is assumed that $A(p)$ has no purely imaginary roots, although the result in this case is otherwise evident. Further it is assumed that $A(p)$ goes to zero as $|p|$ becomes infinite, and that no negative resistance elements are included in the amplifier. Another point which should be mentioned is that the analysis does not apply to the stability in any conjugate paths that may exist. This point may be exemplified by the balanced tube or push-pull amplifier, in the normal transmission path of which the tubes of a stage act in series. When the series output is connected back to the series

plex quantity in general, any value which it may take is representable as a point on a plane—the p-plane of Fig. 2. Since only values of p with positive real parts concern us, attention may be confined to the right-hand half of the p-plane. Now draw a closed contour C in the right-hand half of the p-plane which encloses the root p_k. It is evident

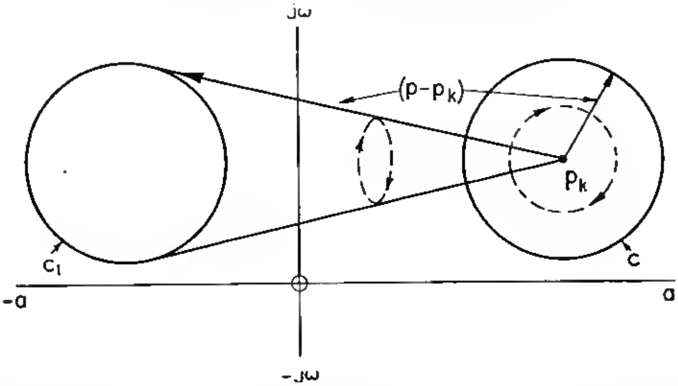

Fig. 2—Plot of two contours C and C_1 in the p-plane. C encloses the root p_k while C_1 does not enclose p_k. The vector $p - p_k$ covers 360 degrees as p traverses C, and covers the net angle of zero as p traverses C_1.

upon inspection of Fig. 2 that the vector extending from the root p_k to the contour makes a complete revolution (360 degrees) in following the closed path. If the contour does not enclose the root, however, as for C_1, then it is clear that when the vector from the root to the contour traverses the whole contour, the net angle turned through by the vector is zero. In the region under consideration we may write

$$f(p) = (p - p_k)\phi(p),$$

where $\phi(p)$ has no zeros within the contour. Hence, when p traverses a closed path and $(p - p_k)$ turns through 360 degrees or through zero the same angle is covered by $f(p)$. If for some different contour several roots are enclosed, it may be shown that $f(p)$ turns through one complete revolution for each of the enclosed roots when p traverses the contour.

In the form in which these considerations are stated, they are not suitable to practical application since complex values of p are involved. Ordinarily, of course, only imaginary values ($p = j\omega$) are conveniently accessible to us since it is a comparatively simple matter to

input, stability of the resultant loop has in general no bearing upon the stability of the path formed with the two tubes of each stage in parallel, since the series and shunt paths are conjugate to one another. To establish the stability of the shunt or parallel path, the transfer factor for that path must be separately determined. In general, the stability criterion applies only to the particular loop investigated, and not to any other existent loop.

measure the response with a sinusoidal impressed wave, but it would involve great difficulties of experiment as well as of interpretation to determine the response with negatively damped waves corresponding to values of p in the right-hand half of the plane. However, these results may be brought within the field of practical experience by a procedure widely used for the purpose.

To include all roots in the right-hand half of the p-plane, the contour must be taken of infinite extent. The path ordinarily followed for this purpose extends from the value $+R$ to $-R$ on the imaginary axis, and is closed by a semicircle of radius R, where R is assumed to expand without limit. It may be noted that in actual amplifier circuits the transfer factor becomes zero when $|p|$ becomes infinite, so that $A(p)$ is zero along the semicircular part of the closed contour. Consequently, the only values of $A(p)$ which differ from zero are those corresponding to finite values of p, along the imaginary axis. In other words, the plot of $A(p)$ under these conditions comes down to the plot of $A(j\omega)$ where ω is finite. Hence, if we plot $A(j\omega)$ for all values of ω from minus to plus infinity, there will be no roots with positive real parts and the system will be stable when the vector from (1, 0) to the curve sweeps through a net angle of zero. The system will be unstable when the vector sweeps through 360 degrees, or an integral multiple thereof.

Two types of transfer factor curves may be considered as illustrations. The first of these shown in Fig. 3 corresponds to that for a re-

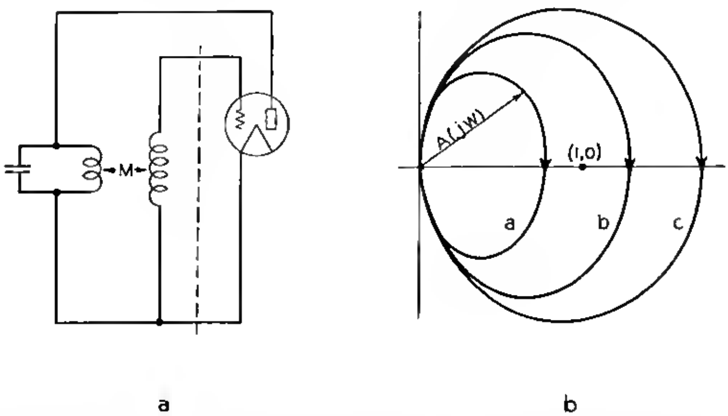

Fig. 3—Schematic of a reversed feed-back oscillator circuit at the left. At the right plot of the transfer factor $A(j\omega)$ around the feed-back loop of Fig. 3a over the frequency range from zero to very high frequencies. The imaginary part of the transfer factor is plotted as ordinate against the real part as abscissa for the three curves a, b, c, which correspond to increasing gains around the loop. Condition a is stable, while b and c are unstable.

versed feed-back oscillator circuit, the three curves marked *a*, *b*, *c*, corresponding to progressively increasing gains around the loop. It will be observed that after the maximum gain has reached and exceeded unity, that the circuit is unstable, since the point (1, 0) is then enclosed. This state of affairs may be contrasted with that existing in the particular form of feed-back circuit to which Fig. 4 applies. Again the

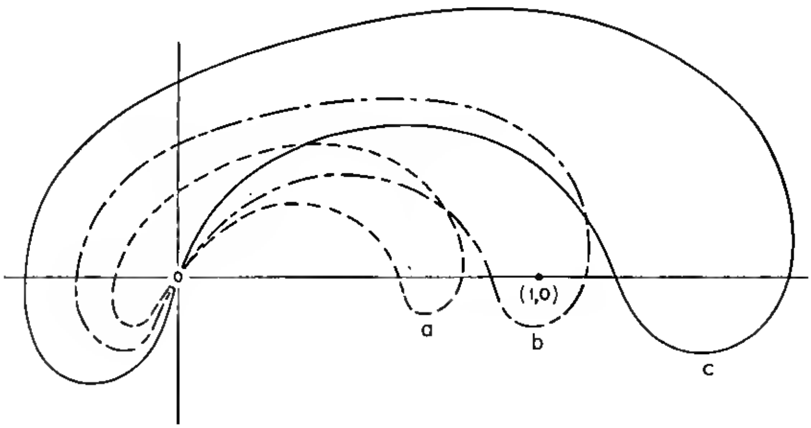

Fig. 4—Transfer factor diagram for a particular form of feed-back circuit, curves *a*, *b*, *c*, corresponding to increasing gains around the feed-back loop. Conditions *a* and *c* are stable, *b* is unstable.

three curves *a*, *b*, *c*, correspond to progressively increasing gains around the feed-back loop. As the gain is increased the system is first stable (*a*), then unstable (*b*), and finally stable (*c*), since it is only within curve (*b*) that the point (1, 0) is enclosed. This striking example is the one which was investigated experimentally. The methods used in determining the transfer factor diagram form the subject of the next section.

Measuring Methods

Application of the Nyquist stability criterion requires the determination of the vector transfer factor around the feed-back loop at all frequencies. This is usually effected by opening the circuit at any point which provides convenient impedances looking in both directions from the break. These points are then connected to an oscillator and to suitable measuring circuits, which are to be described. Care must be taken to ensure that the oscillator and measuring circuit impedances are equal to the output and input impedances respectively of the circuit under test. This precaution is necessary in order that the transfer factor in the measuring condition may not differ significantly from that existing in the operating condition.

Two methods of measurement have been found useful. The first is a null method capable of good precision over a wide frequency range. The second is a visual method in which the transfer factor polar diagram is traced on the screen of a cathode ray oscillograph. This method is not capable of very great precision and, in the model used, the frequency range is somewhat restricted. However, it permits of a rapid survey of the situation for which its precision is adequate, before proceeding with the slower and more precise measurements of the null method, where the latter are required. By making such a preliminary survey the critical frequency ranges can be mapped out for precise measurement, thereby eliminating a large amount of unnecessary labor.

Null Method

In the more precise measurements extending over a wide frequency range, special care is required to ensure freedom from errors in the measurement of phase angles and amplitudes. Much of the difficulty associated with direct measurement over wide frequency ranges is avoided by the use of a simple demodulation scheme. In this scheme, the potentials to be compared are modulated down to a fixed frequency (in actual use 1000 cycles) regardless of the frequency at which the test is being made. In this way a minimum portion of the circuit carries the high frequency. Further this permits the use of voice frequency attenuators, phase shifters, and amplifiers which in fact require calibration at only a single frequency.

In this arrangement, as shown in Fig. 5, demodulators are shunted across the input and output terminals of the circuit under test. A single oscillator supplies the carrier to both demodulators, its frequency differing by 1000 cycles from the frequency supplied to the circuit under test. The demodulated outputs are connected through attenuators and phase shifters to a common amplifier detector. The attenuators and phase shifters are adjusted until the detector gives a null reading. When this condition obtains the difference in the attenuator settings in the two branches is equal to the gain or loss of the circuit under test, and the difference in the phase shifter settings is either equal to or the negative of the phase shift of the circuit under test. To show this, denote the amplifier output voltage by $P_0 \cos (2\pi ft - \phi)$, and the beat frequency voltage supplied to the demodulators by $P \cos 2\pi(f \pm 1000)t$. The demodulated output, proportional to the product of the two applied waves, is then

$$PP_0 \cos (2\pi \cdot 1000t \mp \phi).$$

Correspondingly, the demodulated output from the other demodulator

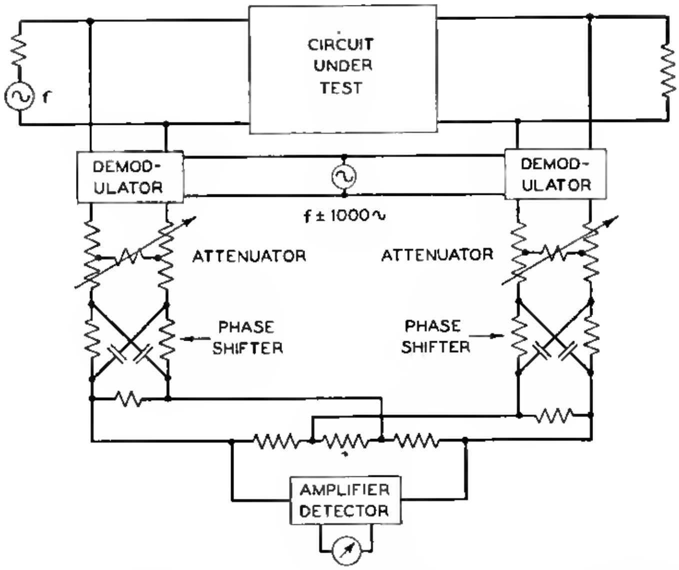

Fig. 5—Schematic diagram of the null method used to measure the transfer factor.

connected across the input is given by

$$PP_i \cos (2\pi \cdot 1000t).$$

If now these two waves are to be made to cancel, there must be a difference in the attenuation of the two branches equal to the ratio P_0/P_i, and a difference in the phase shift equal to $\mp \phi$. The change in sign of the phase angle introduced by setting the beat oscillator above or below the test frequency is most conveniently handled by setting the carrier oscillator consistently on the same side of the test frequency in making a run over the frequency range.

By using a high gain amplifier preceding the detector, the precision may be made great, limited only by circuit noise and by interference. The attenuators and phase shifters are calibrated separately. It should be noted that any difference in the transfer constants of the two demodulator circuits may be compensated by an initial adjustment which is carried out by paralleling the input terminals of the two demodulators across a source of electromotive force. With the particular type of phase shifter used the phase shift may be changed without altering the attenuation, so that the two settings for amplitude and phase may be made independently.

Visual Method

In the visual method of observation, a steady potential proportional to the inphase component of the transfer factor is impressed across one pair of plates of a cathode ray oscillograph and another steady potential proportional to the quadrature component is impressed across the other pair of plates, the constant of proportionality being the same for the two components. In this way the transfer factor at any frequency appears as a single point, the vector from the origin to the displaced beam constituting the transfer factor. The locus of all these points, i.e., vector tips, over the frequency range constitutes the transfer factor polar diagram.

To provide rectified potentials proportional to inphase and to quadrature components respectively, use is made of the properties of the so-called vacuum tube wattmeter.[10] As used in practice, this device consists of two triodes in push-pull connection (Fig. 6), the series arm of the grid circuit being connected to the unknown potential, and the

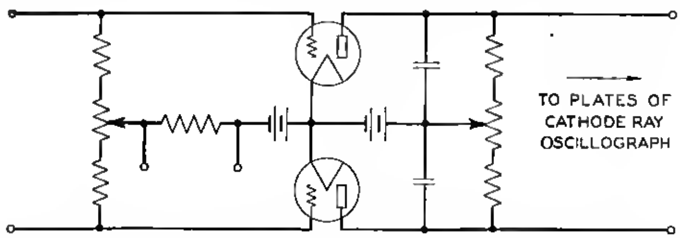

Fig. 6—Circuit of a vacuum tube wattmeter used to provide a rectified potential proportional to the product of the two impressed grid potentials (both of the same frequency) multiplied by the cosine of the phase angle between them.

shunt arm of the grid circuit being connected to a source of the same frequency but of standard phase. Under these conditions the rectified output in the plate circuit flowing in series with the two plates is proportional to the product of the two impressed voltages multipled by the cosine of the angle between them.

As shown in Fig. 7, two separate wattmeters are employed, one for each phase, their series input terminals being connected together across the output of the circuit under test. To the common branch of one of these wattmeters is supplied the same potential as is fed to the input of the circuit under test. The rectified output of this wattmeter therefore is proportional to the product of the input and output voltages multiplied by the cosine of the transfer factor phase angle. This po-

[10] U. S. Patent 1,586,533; Turner and McNamara, *Proc. I. R. E.*, vol. 18, p. 1743; October (1930).

tential is supplied to those plates of the oscillograph which produce a horizontal deflection. To the common branch of the other wattmeter is applied a potential equal in amplitude to the input voltage but lagging behind it by 90 degrees. The rectified output of this wattmeter is proportional to the product of input and output voltages multiplied by the cosine of the transfer factor phase angle minus 90 degrees, or in other words proportional to the sine of the transfer factor phase angle. This voltage is supplied to those plates of the oscillograph which produce a vertical deflection. We have then across one pair of plates of the oscillograph a steady potential proportional to the real component

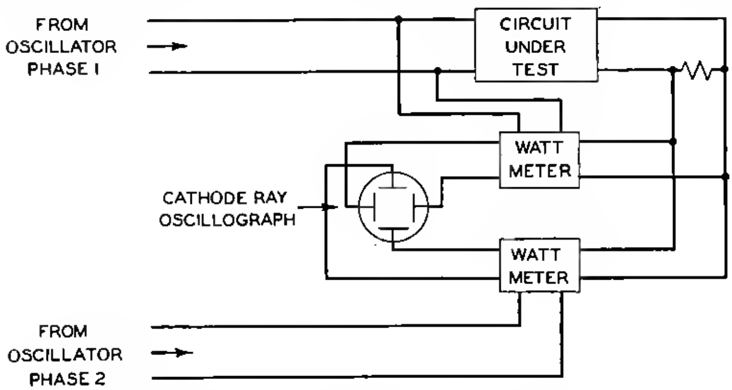

Fig. 7—Schematic diagram of the circuit used to plot the transfer factor diagram on the screen of a cathode ray oscillograph.

of the transfer factor, and across the other pair of plates we have impressed a steady potential proportional to the imaginary component of the transfer factor. These two components act upon the beam of the oscillograph to produce a deflection which in amplitude and in phase is the resultant of the two component deflections and so corresponds to the transfer factor.

It will be observed that the above procedure requires a two-phase source of constant amplitude, the frequency of which is variable over the range necessary to establish the properties of the amplifier. In the present instance the frequency range extends from 0.5 to 30 kilocycles, and the accuracy required is of the order of five per cent.

A schematic of the two-phase oscillator used is shown in Fig. 8. This oscillator is of the heterodyne type. Two independent sources are used, one of constant frequency (100 kilocycles), the other variable in frequency and practically constant in amplitude over the range of 100 to 130 kilocycles. As indicated in the figure, the variable fre-

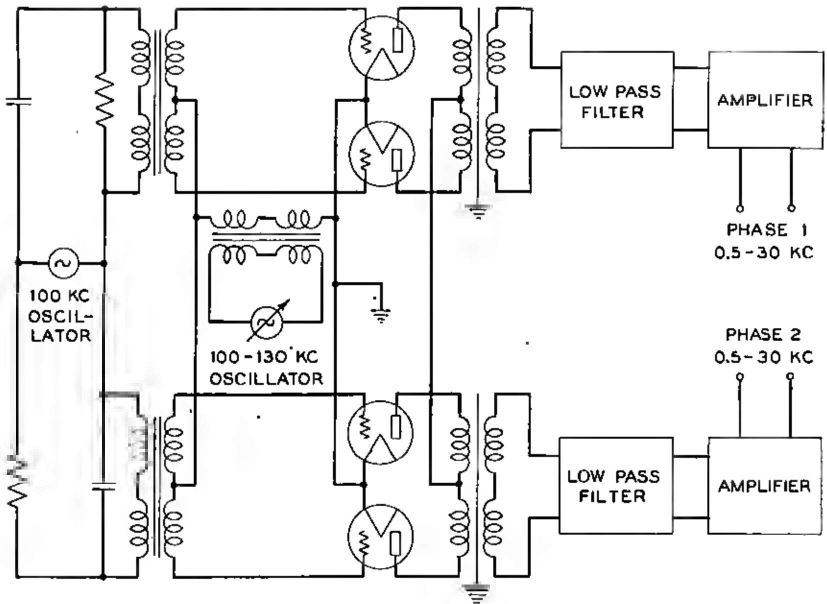

Fig. 8—Circuit diagram of a heterodyne type two-phase oscillator, the output frequency of which is continuously variable from 0.5 to 30 kilocycles. The output of each phase and the 90-degree difference between the two phases are practically constant over the frequency range.

quency oscillator is connected to the common branches of the two push-pull modulators. The fixed frequency oscillator is connected in series with the grid circuits of the two modulators. The resistance-capacity networks shown in the circuits of the fixed frequency oscillator are provided to produce phase shifts of 90 degrees between the two series voltages of the two modulators. In the same manner as that discussed before in connection with the null method measuring circuit, the phase shift introduced to the fixed frequency is maintained in the beat frequency output, so that the phase difference of 90 degrees is preserved in the outputs of the two modulators when the variable frequency oscillator goes from about 100.5 to 130 kilocycles. The outputs of the two phases are connected to the test amplifier and to the wattmeters as shown in the preceding Fig. 7.

Comparison of the Methods

Measurements of transfer factors by the two methods outlined above were found to be in agreement within the error of measurement. The visual method as developed was capable of use over only a very restricted frequency range as compared to the null method, but it covered the region of particular interest in the experiments conducted for the purpose of testing the stability criterion. Through its use, measurements over its frequency range could be made in a few minutes time, whereas corresponding measurements by the more precise null method required three to six hours. Of course the time intervals cited do not include time occupied in setting up and adjusting the apparatus.

Test Amplifier and Experimental Results

Test Amplifier

The stability criterion indicates three distinct conditions of interest, one of which is unstable, the other two being stable. The unstable condition (1) is that in which the transfer factor curve encloses the point (1, 0). Two stable conditions are those in which (1, 0) is not enclosed by the curve, but in which (2) the curve crosses the zero phase shift axis at points greater than unity, and (3) the curve does not cross the zero phase shift axis at points greater than unity. Condition (2) is of particular interest because while it is judged stable on the basis of Nyquist's criterion it would appear to be unstable on the basis of the older transfer criterion discussed in the first and second sections.

For test purposes an amplifier was designed which, upon variation of an attenuator in the feed-back path, would satisfy each of the three above conditions in turn. The amplifier schematic is shown in Fig. 9.

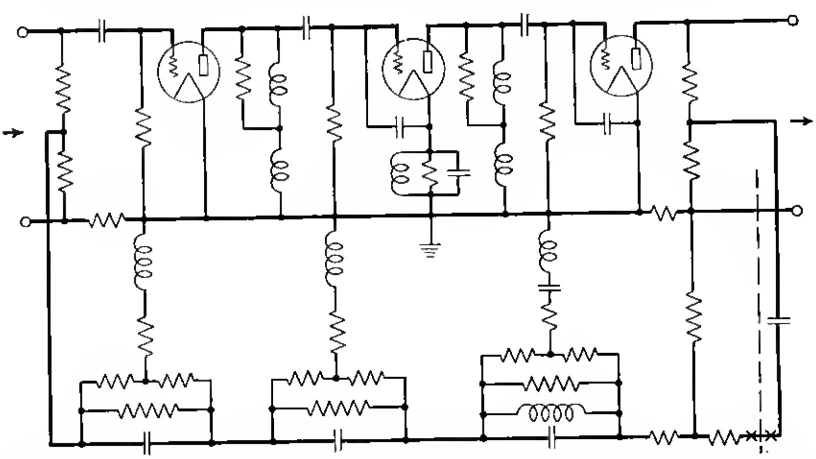

Fig. 9—Circuit diagram of the feed-back amplifier used in testing the stability criterion. The dashed line indicates the point at which the loop was broken for measurement of the transfer factor. At the left of this line is shown the resistance attenuator provided to vary the gain around the feed-back loop.

It has three stages, the first two tubes being space charge grid pentodes, and the last one a triode. The interstage coupling circuits were made up of simple inductances and resistances as shown. The amplifier was designed by E. L. Norton and E. E. Aldrich to provide a transfer factor characteristic having the desired shape, i.e., a loop crossing the zero phase axis in the neighborhood of 10 kilocycles. It will be observed that the feed-back circuit is connected between bridge networks in both input and output circuits, which were provided to eliminate reaction of the input and output circuits upon the feed-back network.[11]

Experimental Results

The transfer factor was measured for a zero setting of the feed-back attenuator over a frequency range of 0.5 to 1200 kilocycles. The results are shown in Fig. 10. The method of plotting this figure requires some discussion. In order to keep the curve within a reasonable size and still show the necessary details the scale has been made logarithmic by plotting the gain around the loop in decibels instead of the corresponding numerical ratios. It is of course impossible to carry this out completely on a polar diagram since the transfer factor goes to zero at high frequencies. To take care of this the scale is made logarithmic only above zero gain, corresponding to unit transfer ratio, and is linear below. It should be noted that if the logarithmic portion of the scale is translated outward so that the zero decibel point lies successively in the regions marked A, B, C, and D, the indicated amplifier conditions correspond to those designated above as (1), (2), (1) and (3) respectively. Experimentally an increase of the feed-back attenuator corresponds to such a translation of the logarithmic scale by an amount equal to the increase in attenuation. Therefore, the transition from one condition to another should occur when the attenuator setting is equal to the gain at a zero phase point in the curve as measured with a zero attenuator setting.

The test of the stability criterion consists of a determination of the attenuator settings at which oscillations begin, and a comparison of these settings with those at which a transition from a stable to an unstable condition is predicted by the theory. Experimentally oscillations were found to occur in regions A and C and not in regions B and D which is in qualitative agreement with Nyquist's predictions. Quantitatively the measured and predicted transition points agreed within one decibel which is estimated to be within the experimental error.

It should be noted that the plotted curve has been drawn up for $A(j\omega)$, no points of $A(-j\omega)$ being shown, although both are required

[11] H. S. Black, *Bell Sys. Tech. Jour.*, January, 1934.

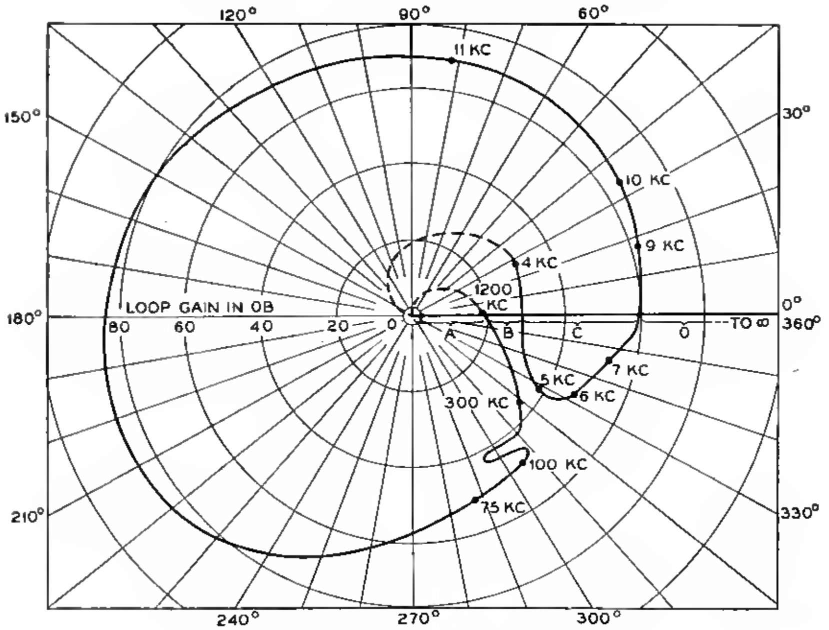

Fig. 10—Transfer factor diagram for the amplifier of Fig. 9 with the feed-back attenuator set at zero decibel.

by the theoretical derivation. Where the transfer factor is zero at zero frequency, only $A(j\omega)$ is required since the loop then closes for positive values of ω. In amplifiers transmitting d.c., however, both positive and negative values of ω are needed[8] to form a closed loop. In any case $A(-j\omega)$ is the mirror image of $A(j\omega)$ about the x-axis.

Extensions of the Criterion

Nonlinear Amplifier

The stability criterion which was verified by the experiments reported in the preceding section is framed for linear systems, those in which the steady state response is linearly proportional to the applied force. In vacuum tube circuits, linearity is best approximated at small force amplitudes, and is departed from to an extent dependent upon the impressed potentials, as well as upon tube and circuit characteristics. The divergence from linearity becomes well marked when the load capacities of the tubes are approached, or when grid current is made to flow through large grid impedances. The question then arises as to the form which the stability criterion takes when a tube circuit is

[8] Loc. cit.

operated in a nonlinear region—let us say by impressing upon the circuit a sufficiently large alternating potential provided by an external independent generator.

To answer this question we may consider the response of the amplifier, loaded by the independent generator, to a small alternating potential introduced for test purposes. Since the response of the system is known to be linear from the theory of perturbations, we might attempt to apply the linear criterion to the small superposed force. To do this it is necessary to measure the transfer factor for the small superposed force over the frequency range at a particular load of interest.

Application of the experimental technique to this extended criterion introduces difficulties since the opening of the feed-back loop for measuring purposes disturbs to a certain extent the distribution of these loads, particularly the harmonics, and modulation products in general. This makes it difficult to get the same loading effect when the loop is opened for measuring purposes as obtained when the loop is closed. Another consideration is that the response to the small component may be expected to vary in general at different points on the loading wave, so that the measuring procedure averages the response over a cycle of the loading wave. A method of measurement analogous to that of the flutter bridge would be required to evaluate the transfer factor at points of the loading cycle. Further, the measuring apparatus is affected by the presence of the loading currents when these are sufficiently large. In the present case in which the loading frequency (60 kilocycles) was far removed in the frequency scale from the test frequencies, it was found possible to approximate the necessary measurements by the insertion of selective circuits.

The curves of Fig. 11 represent portions of the transfer factor polar diagram for an amplifier similar to the one previously described, measured by the visual method with different loading amplitudes. The effect of the load on this particular amplifier is to change both phase shift and amplitude so that the curves shrink both radially and tangentially, pulling the loop back across the zero phase axis until, at the heaviest load, the two low-frequency crossings are completely eliminated. If the extended criterion is valid, we should expect the amplifier to be stable at any setting of the feed-back attenuator. As the load is decreased from this value, the crossings occur at successively higher gains so that the start of oscillations would occur at progressively higher settings of the feed-back attenuator.

The curves of Fig. 12 show the attenuator settings predicted by the extended criterion and those determined by direct observation of the attenuator setting required for oscillations when the feed-back circuit

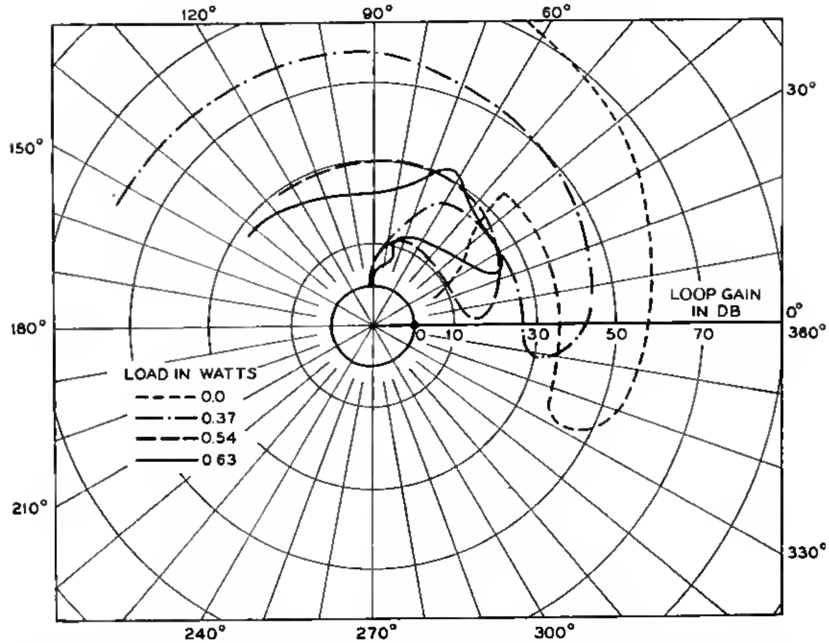

Fig. 11—The transfer factor diagram for the amplifier of Fig. 9 with the feed-back attenuator set at zero decibel. The four curves shown correspond to different amounts of the 60-kilocycle load.

was closed. Two sets of curves are shown, one for each of the low-frequency crossings. These are plotted against the loading amplitude. The agreement between the experimental and predicted values is close for the higher gain crossing at small loading amplitudes, but a divergence is apparent at high loads. For the lower crossing there is a divergence of 1.5 decibels at low loads, which changes sign and becomes greater at the higher loads. These divergences may be ascribed to a variety of causes among which probably the most important are the effects of harmonics upon the amplifier loading, overloading of the measuring apparatus by harmonics of the loading electromotive force, and phase shifts introduced by the selective circuits. The last two causes may be eliminated by improved technique, but the first cause in general introduces a fundamental difficulty, particularly important when large nonlinearities are involved.

Negative Impedances

One of the early forms of stability criterion mentioned in the first section was that relating to the measured impedance of the circuit.

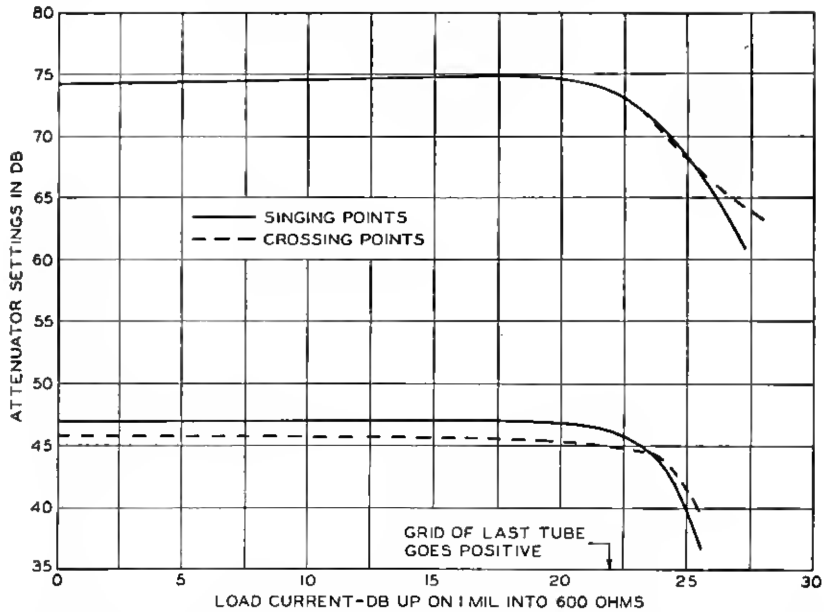

Fig. 12—Comparison of feed-back attenuator settings required for the starting of oscillations, and those deduced from the transfer diagram, plotted as functions of the 60-kilocycle load. The two dashed curves correspond to the two points (roughly 4.5 and 8 kilocycles) at which the transfer factor diagram (Fig. 11) crosses the zero phase axis. The gains of Figs. 11 and 12 cannot be compared directly because of a change made in the amplifier circuit of Fig. 12 which increased the loop gain.

Nyquist's criterion involving the transfer factor may be transformed so as to formulate a more complete criterion involving such an impedance.

To do this we have to express the factor $(1 - A)$, on which the stability criterion was based, in terms of the circuit impedances. For illustrative purposes we may quote the results obtained with the two fundamental forms of feed-back circuits, the series and shunt types.[12] These results, while obtained for the input circuit of the amplifier, are valid for any other point of the feed-back loop. Further, combinations of the shunt and series type feed-back circuits may be used.

Series Feed-Back

The series circuit is shown in Fig. 13, so called because the feed-back is applied in series with the amplied electromotive force and the amplifier input. The passive impedances marked are those existing when the feed-back loop is broken and terminated as indicated by the

[12] Crisson, *Bell Sys. Tech. Jour.*, vol. X, p. 485.

dotted lines. By direct circuit analysis, the current and voltage amplitudes in the feed-back condition are related by

$$E = (Z + Z_0 + Z_i)(1 - A)I,$$

where A and the Z's are functions of frequency. The total effective circuit impedance is obtained as the multiplier of I in the right member.

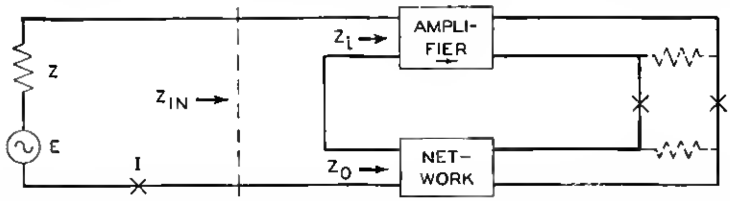

Fig. 13—Series type feed-back circuit. The dotted resistances indicate the terminations applied when the feed-back circuit was broken, to which the passive impedances (Z_iZ_0) apply. Z_{in} represents the effective input impedance with the feed-back circuit connected through.

ber. Subtracting the generator impedance Z from the total, the input impedance becomes

$$Z_{IN} = (Z_0 + Z_i)(1 - A) - AZ,$$

from which;

$$1 - A = \frac{Z}{Z + Z_0 + Z_i}\left(1 + \frac{Z_{IN}}{Z}\right).$$

Of the two factors of the right member, the first one, involving passive impedances alone, can have no roots with positive real part. Any such roots must, therefore, be contained in the second bracketed factor and then only when Z_{in} is negative. Hence paraphrasing the transfer factor criterion, if we plot $-Z_{in}/Z$ over the frequency range, the circuit is stable when the point (1, 0) is not enclosed by the resultant curve.

Shunt Feed-Back

Proceeding as in the series case with the circuit of Fig. 14 we get

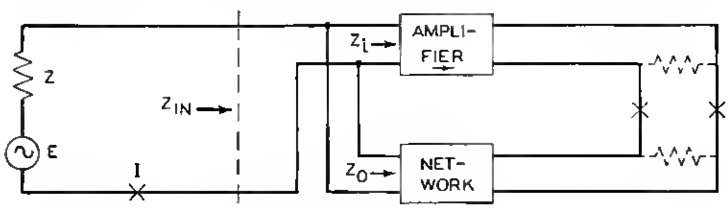

Fig. 14—Shunt type feed-back circuit. The notation corresponds to that of Fig. 13.

$$1 - A = \frac{Z_a}{Z + Z_a}\left(1 + \frac{Z}{Z_{in}}\right),$$

where Z_a represents the impedance of Z_0 and Z_i in parallel. Again only the bracketed term can yield undamped transients so that the criterion involves plotting $-Z/Z_{in}$ over the frequency range; if the resultant curve does not enclose (1, 0) the circuit is stable.

It may be remarked that these results are applicable to circuits including two-terminal negative impedances such as the oscillating arc and the dynatron, which are of the series and the shunt type respectively.

Acknowledgments

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